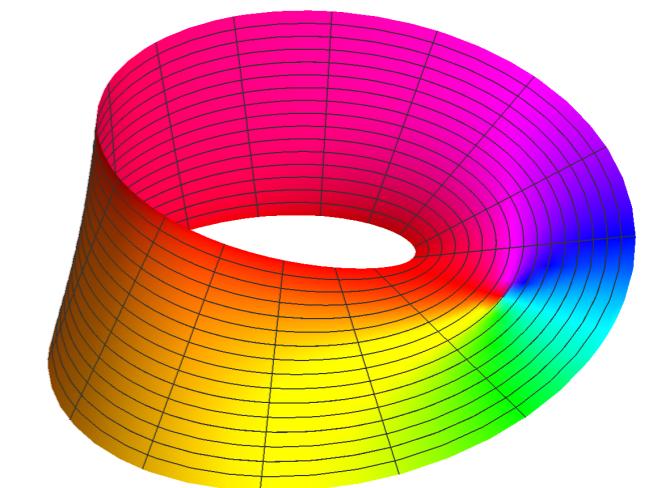
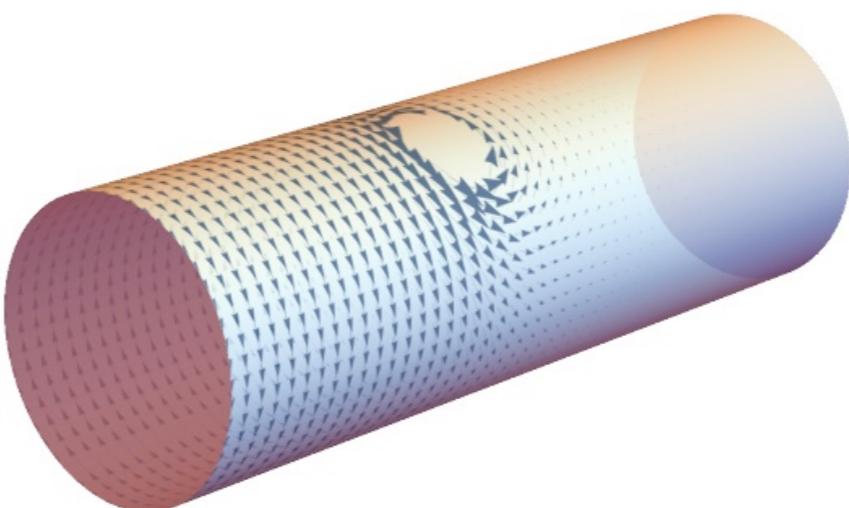
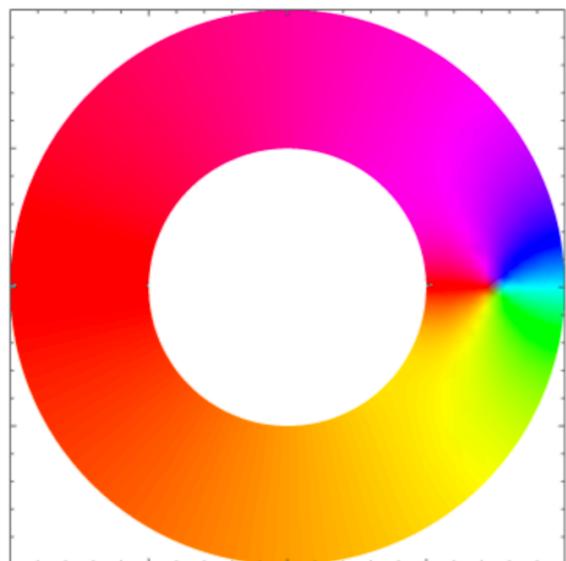


# Superfluid vortex dynamics on peculiar surfaces

Pietro Massignan



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Fotòniques

# Outline

- ◆ Classical vs. superfluid turbulence
- ◆ Perfect fluids in 2D
- ◆ Vortices on an annulus
- ◆ Conformal mappings
- ◆ Vortices on a cylinder won't stand still
- ◆ Cones, Möbius strips, Riemann surfaces, airplane wings, ...

# Collaborators



Nils Guenther

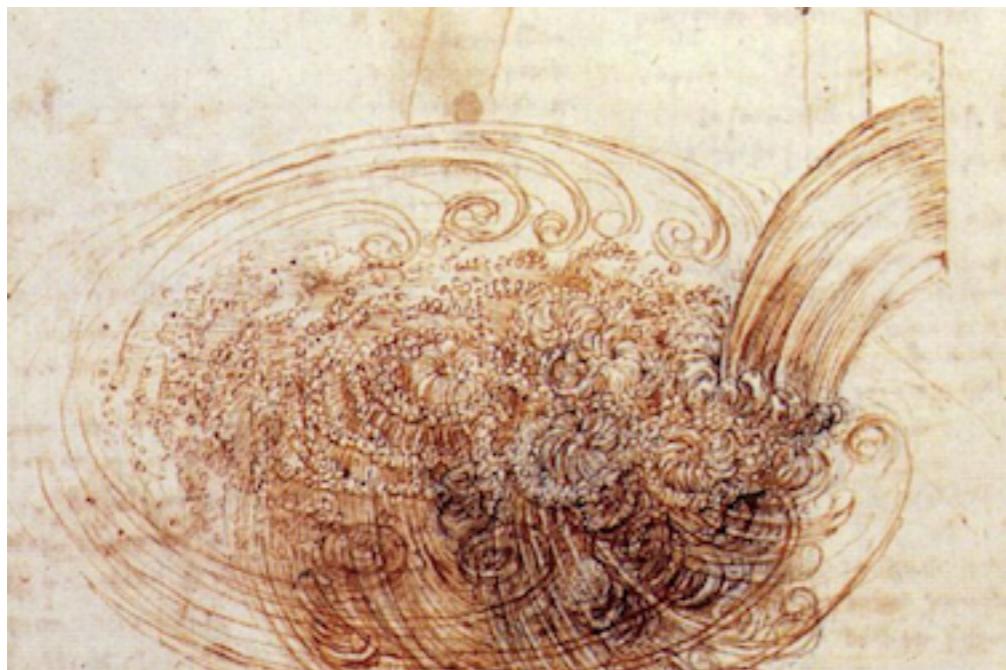


Alexander Fetter



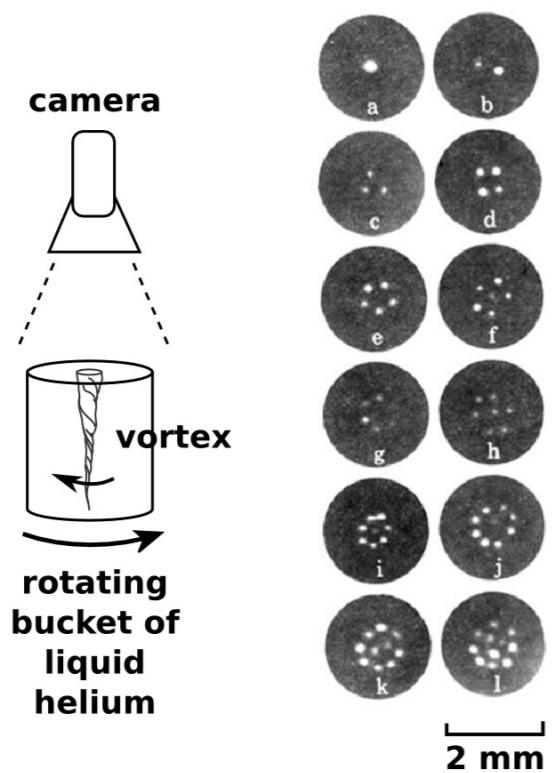
N. Guenther, P. Massignan, and A. Fetter  
Phys. Rev. A **96**, 063608 (2017).

# Classical turbulence

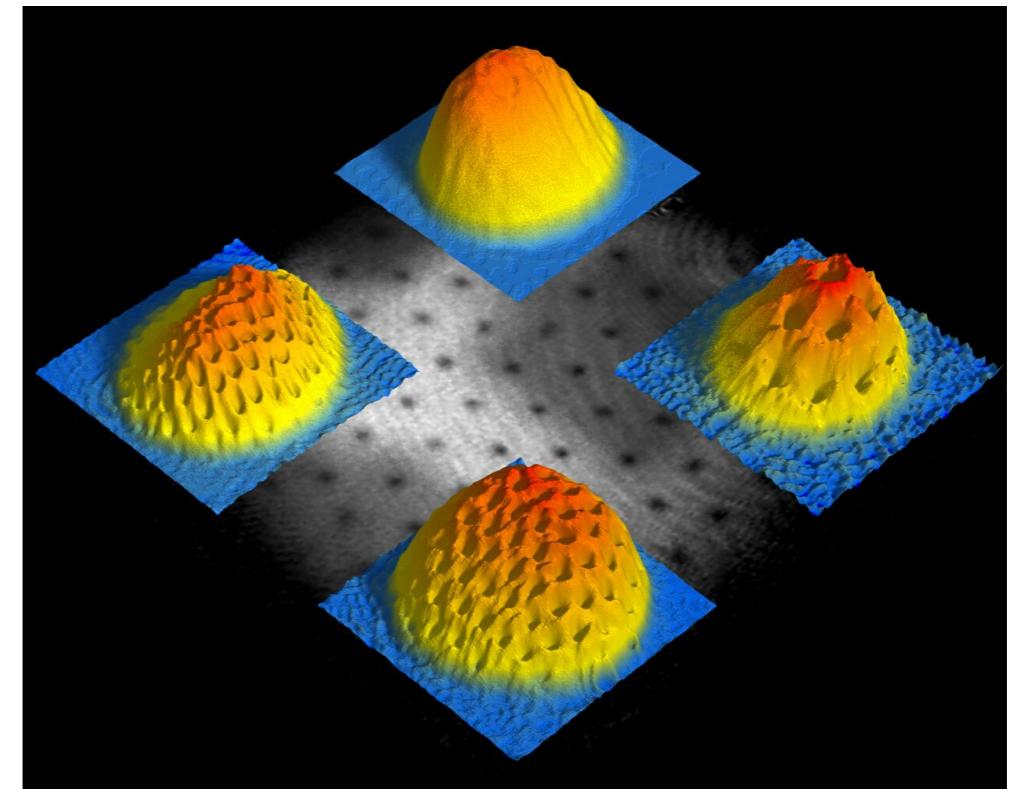


viscous  
multiscale  
chaotic

# Superfluid vortices



[Yarmchuk, Gordon and Packard, 1979]



[Ketterle's group @ MIT, 2001]

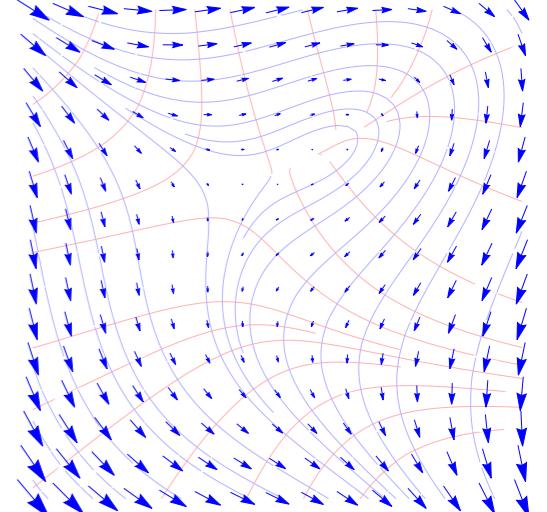
inviscid  
smooth  
crystal vortex patterns

# Superfluid hydrodynamics

- ◆ Macroscopic condensate wavefunction:  $\Psi = \sqrt{n}e^{i\Phi}$
- ◆ Superfluid velocity:  $\mathbf{v} = \frac{\hbar}{M}\nabla\Phi$
- ◆ Vorticity:  $\nabla \times \mathbf{v} = \frac{\hbar}{M}\nabla \times \nabla\Phi = 0$       (*irrotational*)
- ◆ Quantized circulation:  $\oint d\mathbf{l} \cdot \mathbf{v} = \frac{\hbar}{M} \oint d\mathbf{l} \cdot \nabla\Phi = 2\pi j \frac{\hbar}{M}, \quad j \in \mathbb{Z}$
- ◆ Current conservation:  $\frac{dn}{dt} + \nabla \cdot (n\mathbf{v}) = 0$
- ◆ Thomas-Fermi regime → constant  $n$  (*incompressible*):  $\nabla \cdot \mathbf{v} = 0$
- ◆ SF (non-viscous & irrotational) + incompressible = **perfect fluid**

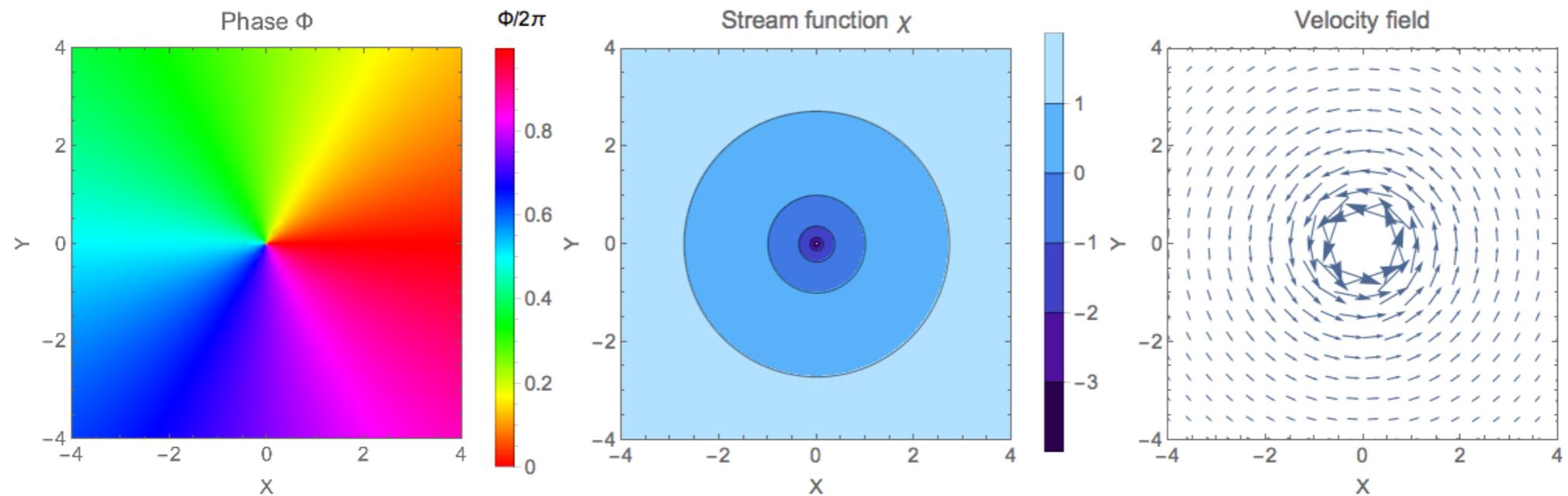
[A. Fetter, Rev. Mod. Phys. **81**, 647 (2009)]

# 2D potential flow

- ♦ For 2D incompressible fluids,  $\mathbf{v} = \left( \frac{\hbar}{M} \right) \hat{\mathbf{n}} \times \nabla \chi$   
 and the velocity is parallel to iso-contours of  $\chi$   
 and orthogonal to iso-contours of  $\Phi$
  - ♦ Perfect fluids in 2D fully described by a *complex potential*  $F = \chi + i\Phi$
  - ♦  $F$  is an analytic function of  $Z = X + iY$
  - ♦ Cauchy-Riemann conditions readily imply:  $v_Y + iv_X = \frac{\hbar}{M} \frac{\partial F}{\partial Z}$
- stream function
- 
- $v_X \propto \frac{\partial \Phi}{\partial X} = -\frac{\partial \chi}{\partial Y}$   
 $v_y \propto \frac{\partial \Phi}{\partial Y} = \frac{\partial \chi}{\partial X}$

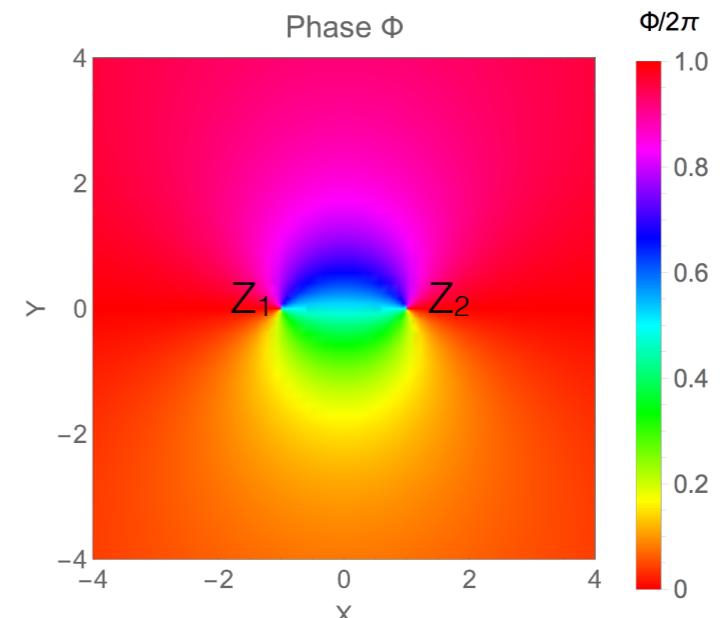
# Vortices on a plane

- ◆ A single vortex at  $Z_0$  :  $F(Z) = \log(Z - Z_0)$



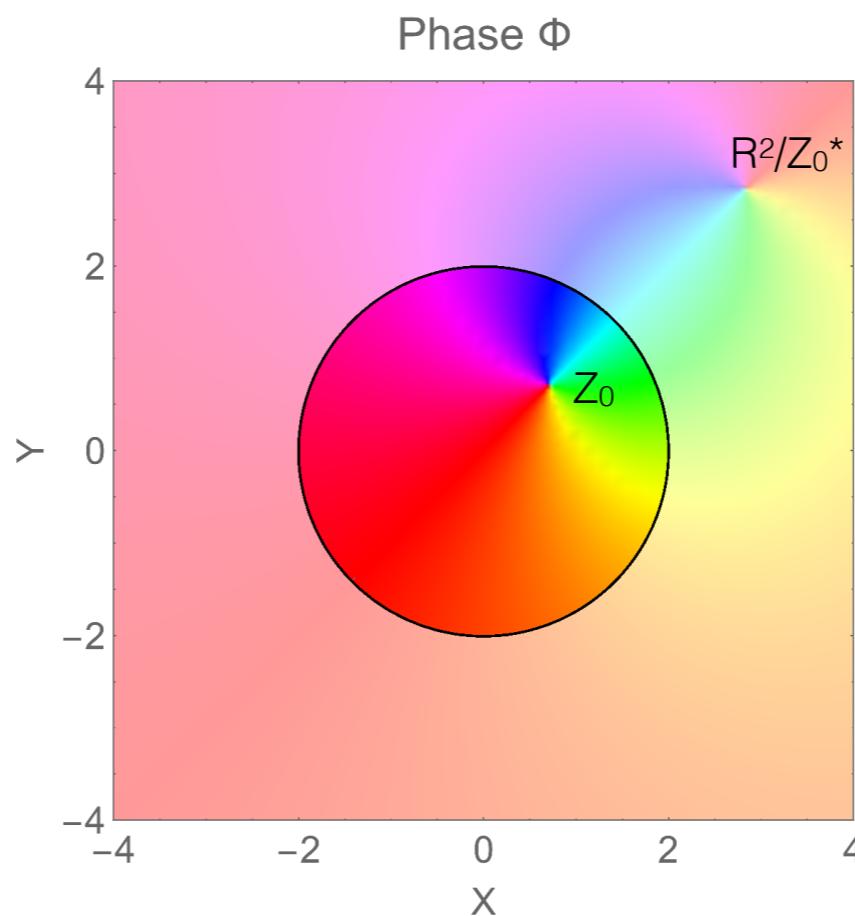
- ◆ A vortex dipole:

$$F(Z) = \log(Z - Z_1) - \log(Z - Z_2)$$



# Surface with boundaries

- ◆ As in electrodynamics, use the method of images
- ◆ Single vortex on a disk of radius  $R$ :  $F(Z) = \log \left( \frac{Z - Z_0}{Z - R^2/Z_0^*} \right)$

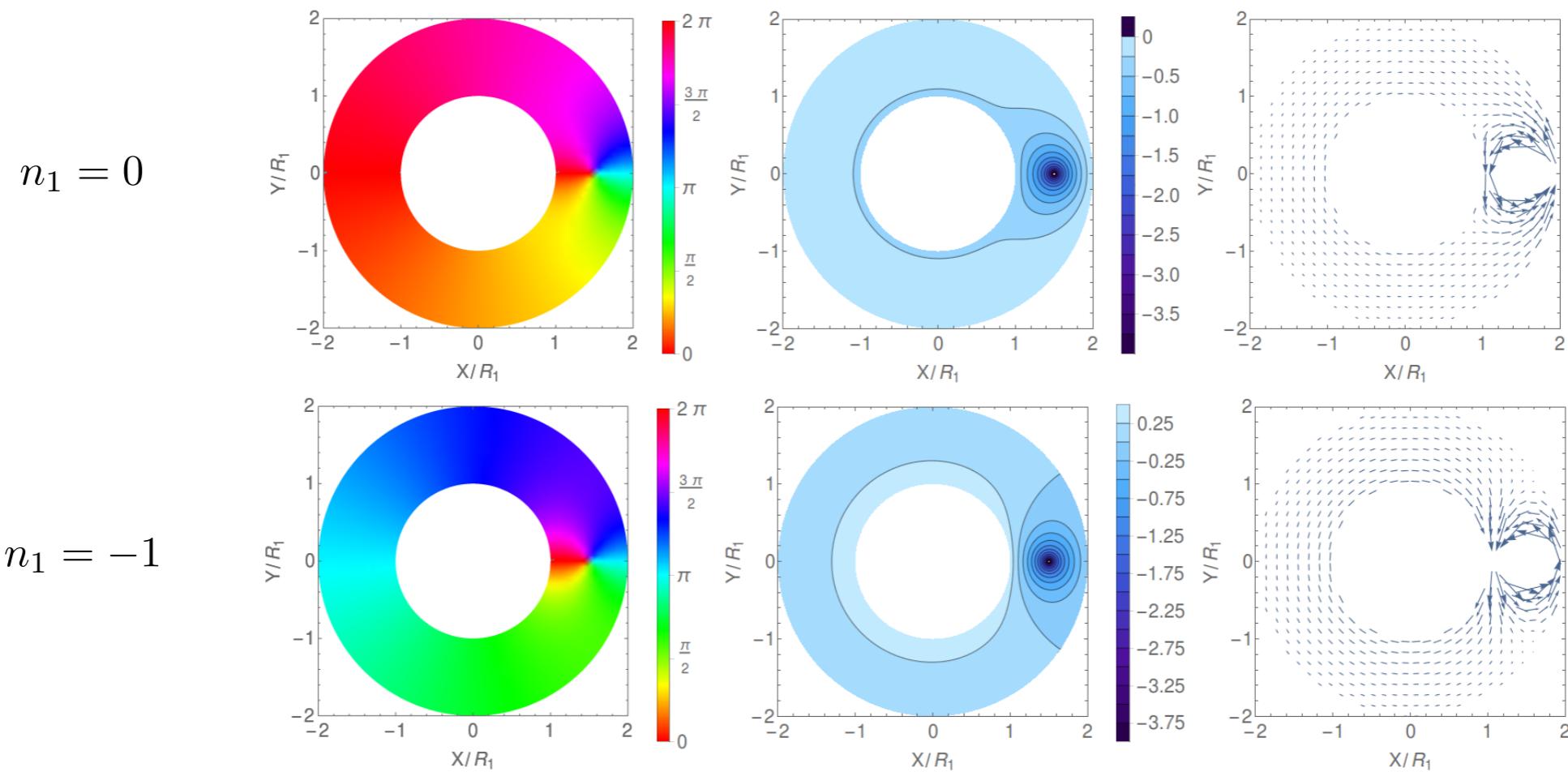


# Vortex on an annulus

- ◆ An annulus has two boundaries → infinite series of images needed

$$◆ \text{ Potential: } F(Z) = n_1 \ln \left( \frac{Z}{R_2} \right) + \ln \left[ \frac{\vartheta_1 \left( -\frac{i}{2} \ln \left( \frac{Z}{Z_0} \right), \frac{R_1}{R_2} \right)}{\vartheta_1 \left( -\frac{i}{2} \ln \left( \frac{ZZ^*_0}{R_2^2} \right), \frac{R_1}{R_2} \right)} \right]$$

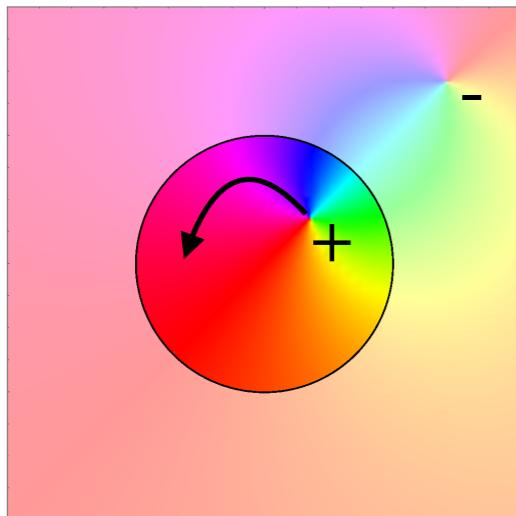
1<sup>st</sup> Jacobi  
Theta  
function



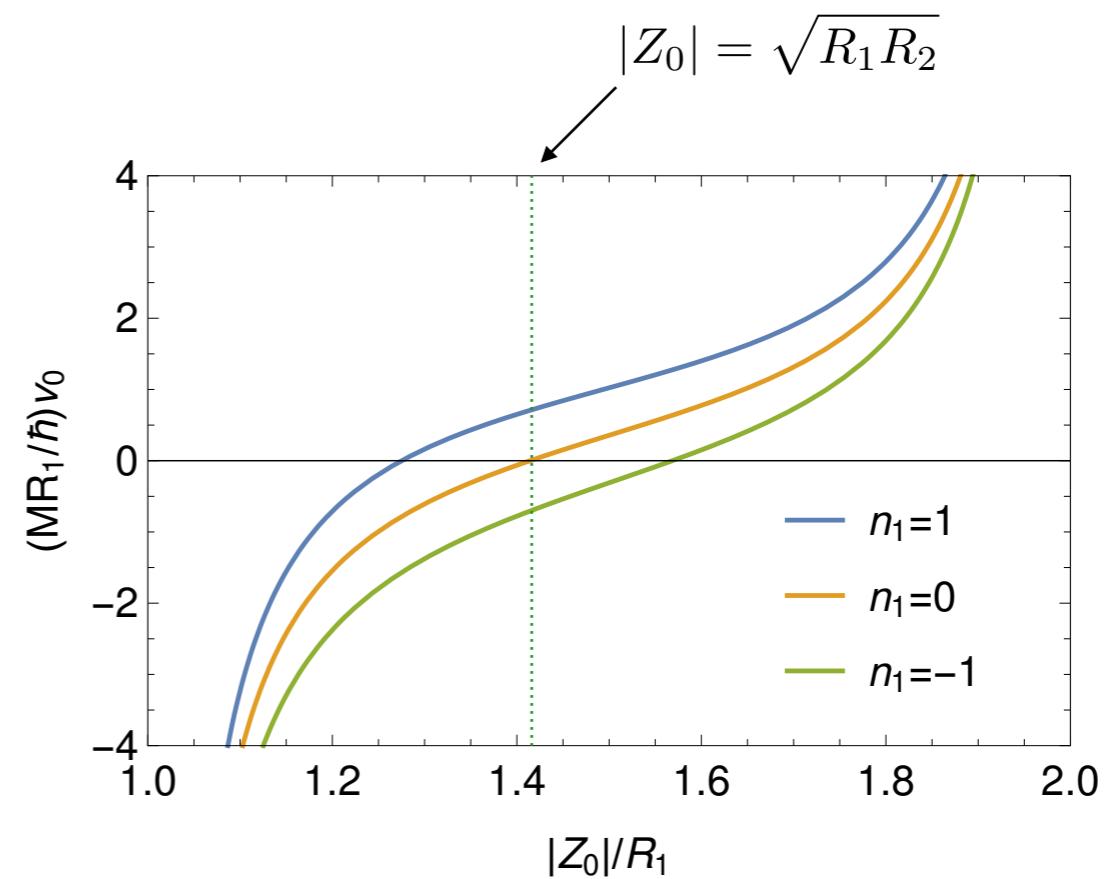
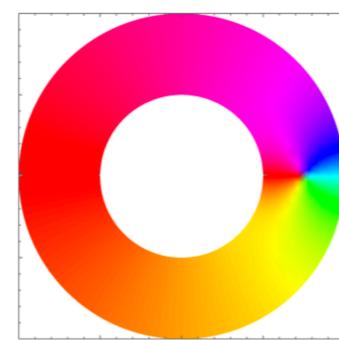
# Velocity of the vortex core

- ◆ A vortex moves with the local flow velocity:

$$\dot{y}_0 + i\dot{x}_0 = \frac{\hbar}{M} \lim_{z \rightarrow z_0} \left[ F'(z) - \frac{1}{z - z_0} \right]$$

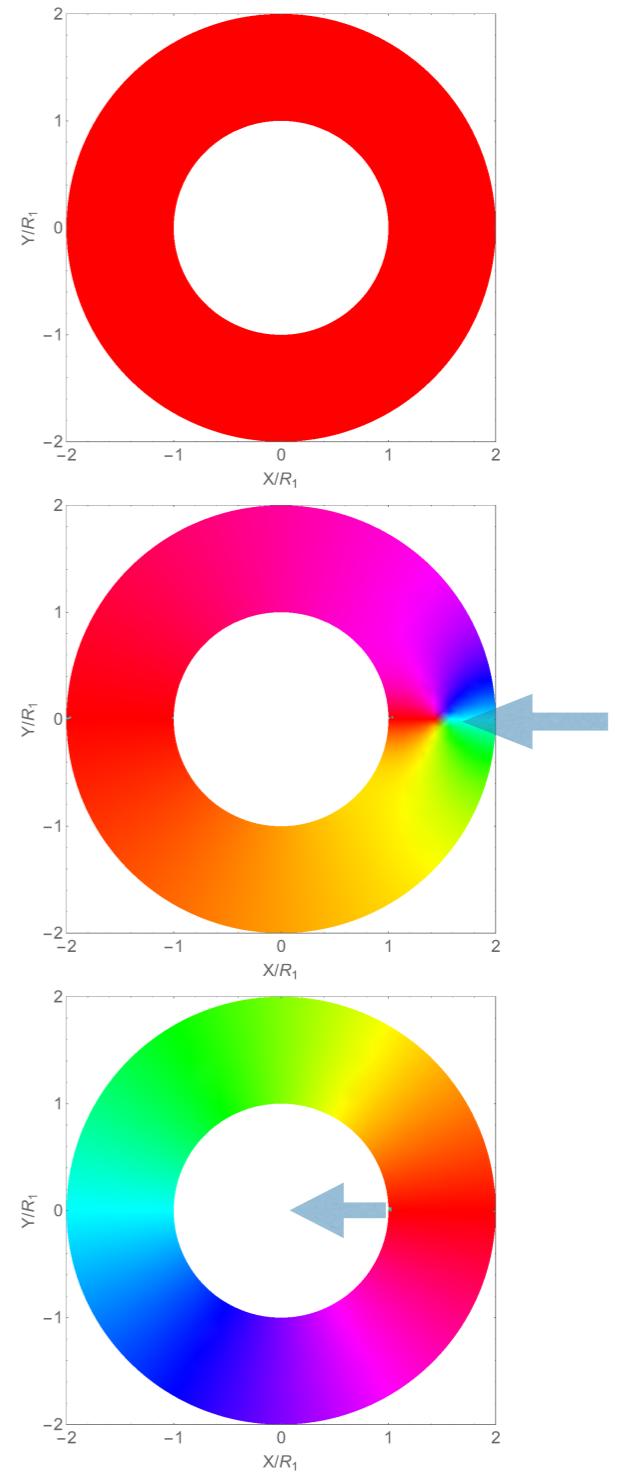


- ◆ Annulus with  $R_2 = 2R_1$ :



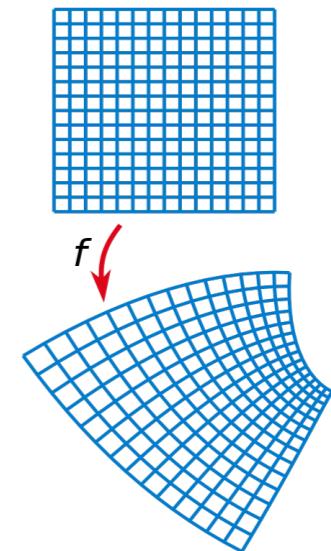
# Laughlin pumping

- ◆ Start with the fluid at rest
- ◆ Stir the fluid from outside at a constant rate
- ◆ A vortex appears on the outer edge, and moves inward
- ◆ The fluid (on average) rotates for  $|Z| > |Z_0|$ , but it remains stationary otherwise
- ◆ As the vortex crosses the inner edge, stop stirring
- ◆ The fluid is left with exactly  $\hbar$  units of angular momentum per particle

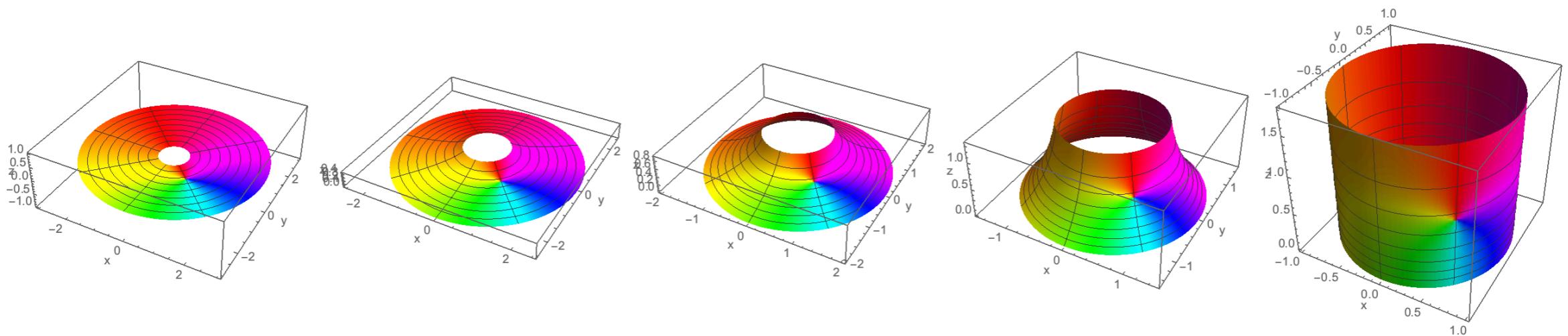


# More complex surfaces?

- ◆ Conformal map:  $f : U \rightarrow V$  conserving angles, and shapes of infinitesimal objects



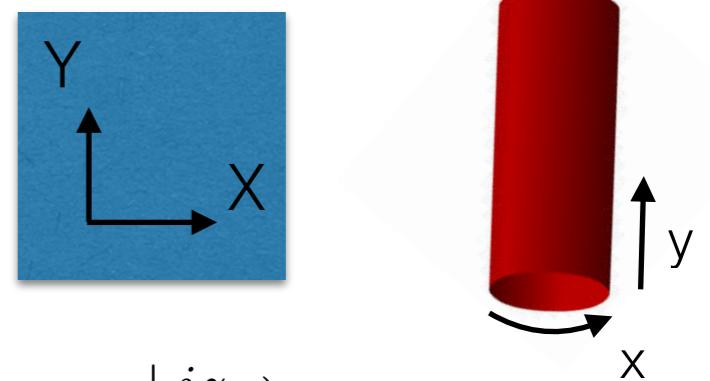
- ◆ The conformal image of a physical flow pattern is still a physical pattern



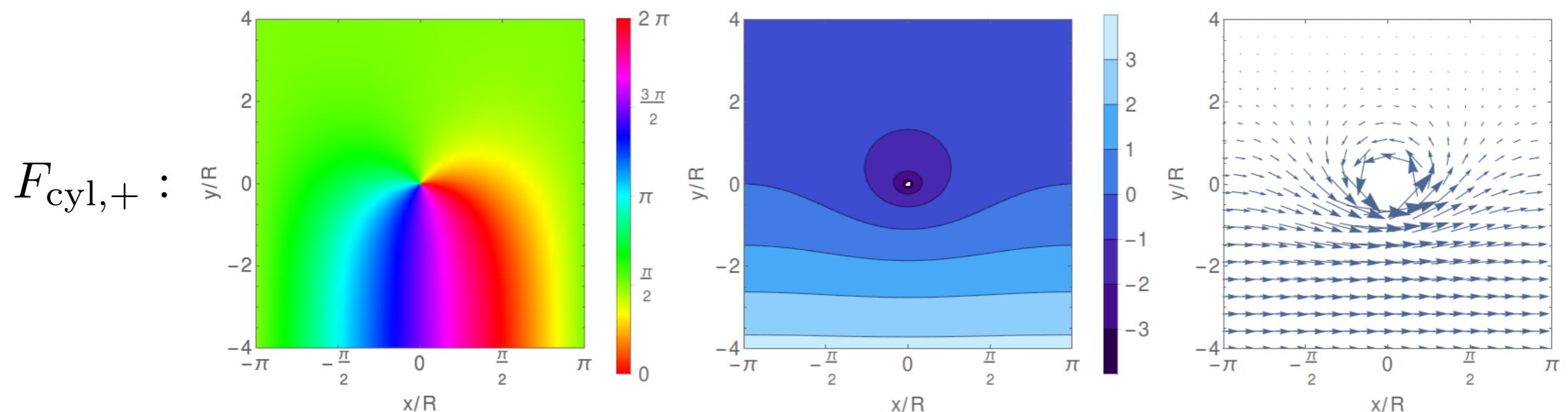
[Turner, Vitelli and Nelson, Rev. Mod. Phys. **82**, 1301 (2010)]

# Vortex on a cylinder

- ◆ Maps linking plane to cylinder:  $Z = e^{\pm i z}$



- ◆  $F_{\text{plane}}(Z) = \ln(Z - Z_0) \rightarrow F_{\text{cyl},\pm}(z) = \ln(e^{\pm iz} - e^{\pm iz_0})$



- ◆ Velocity of the vortex core:  $v_x = \pm \frac{\hbar}{2MR}$

vortices  
on a cylinder  
will *not* stand still

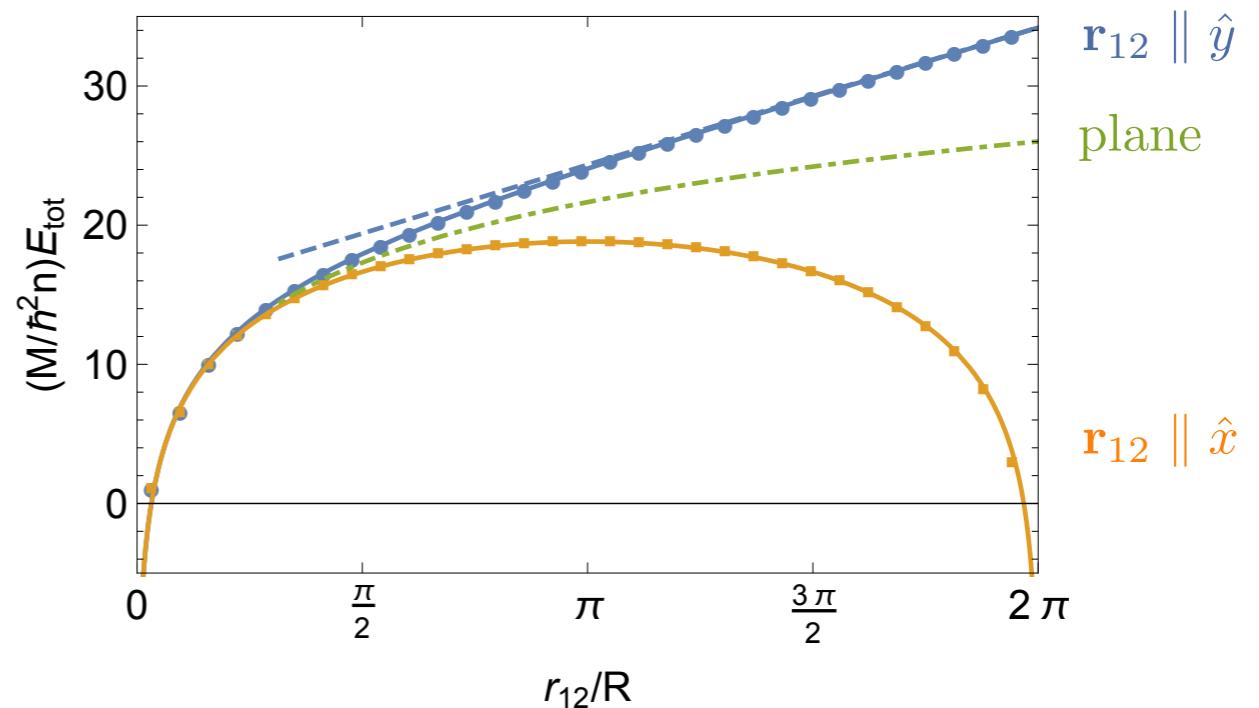
# Energy of the fluid

- ◆ Stream function of  $N$  vortices:  $\chi(\mathbf{r}) = \sum_{i=1}^N q_i \chi(\mathbf{r} - \mathbf{r}_i)$
- ◆ Energy: 
$$\begin{aligned} E_{\text{tot}} &= \frac{nM}{2} \int d^2r |\mathbf{v}(\mathbf{r})|^2 \\ &= \frac{n\hbar^2}{2M} \int d^2r |\nabla \chi(\mathbf{r})|^2 \\ &= \frac{\pi\hbar^2 n}{M} \left[ N \ln \left( \frac{2R}{\xi} \right) + \sum_{i < j}^N q_i q_j \chi(\mathbf{r}_{ij}) \right] \end{aligned}$$

- ◆ Energy of a vortex dipole:

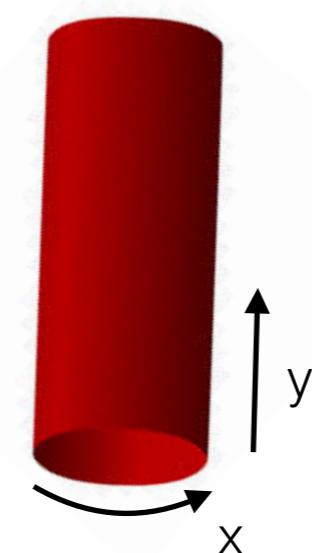
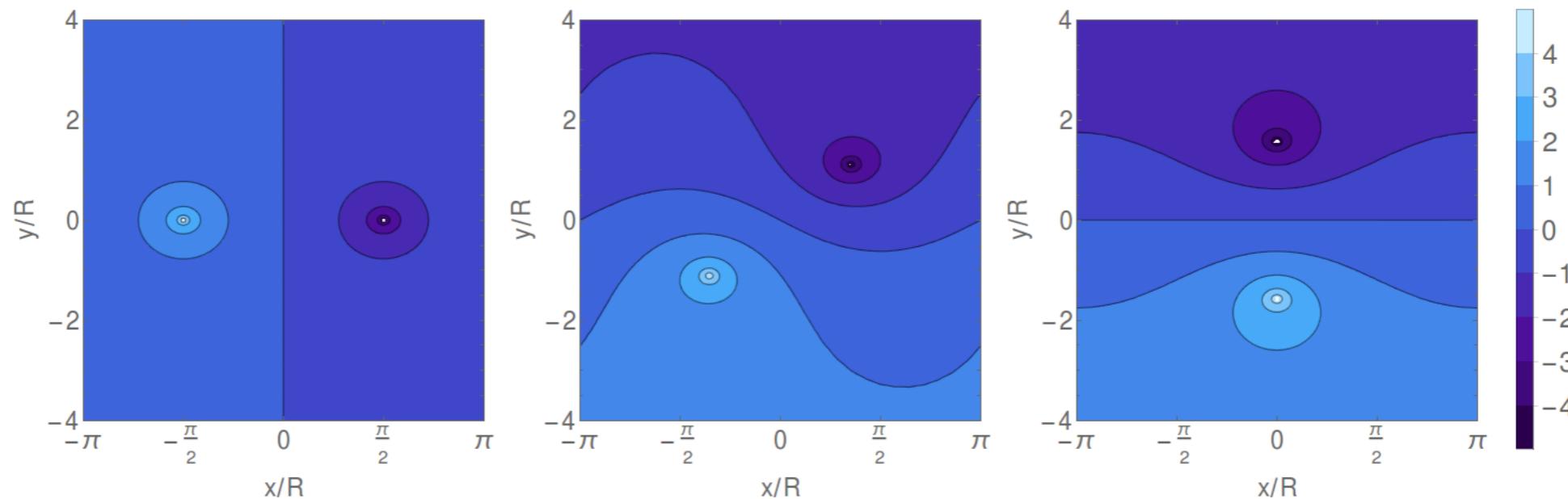
grows linearly for  $r_{12} \gg R$

**BKT physics  
will *not* happen  
on a cylinder**



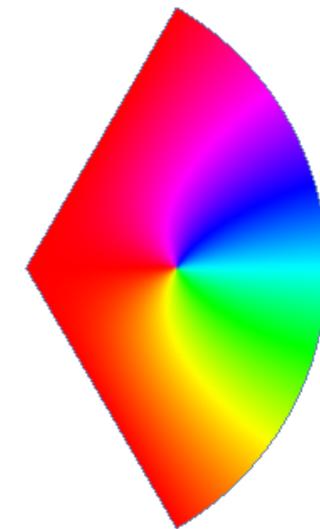
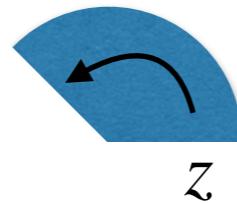
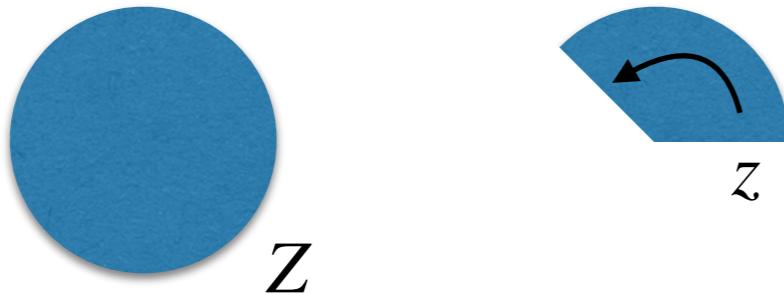
# Motion of a vortex dipole

- ◆ Different trajectories, depending on the orientation of the dipole axis:

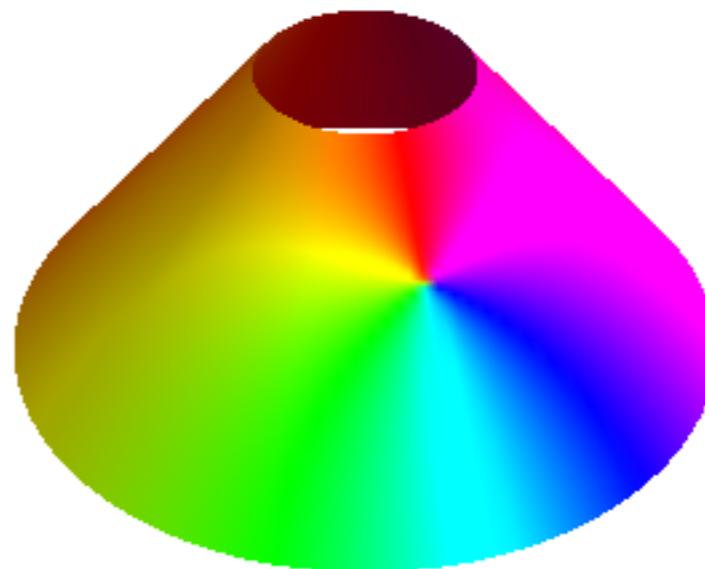


# Vortices on a cone

- ◆ Conformal map from plane to sector of aperture  $2\pi/\alpha$ :  $Z = z^\alpha$  ( $\alpha > 1$ )

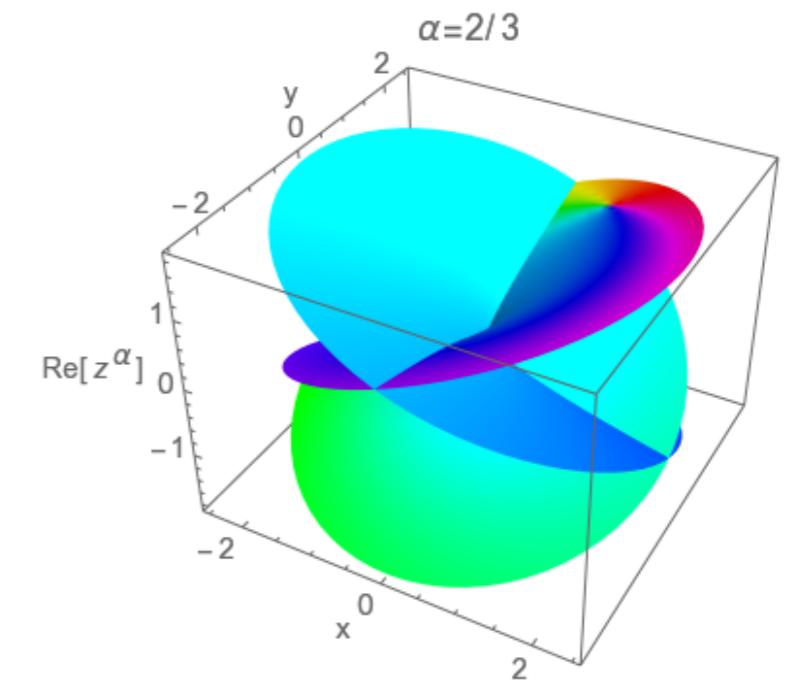
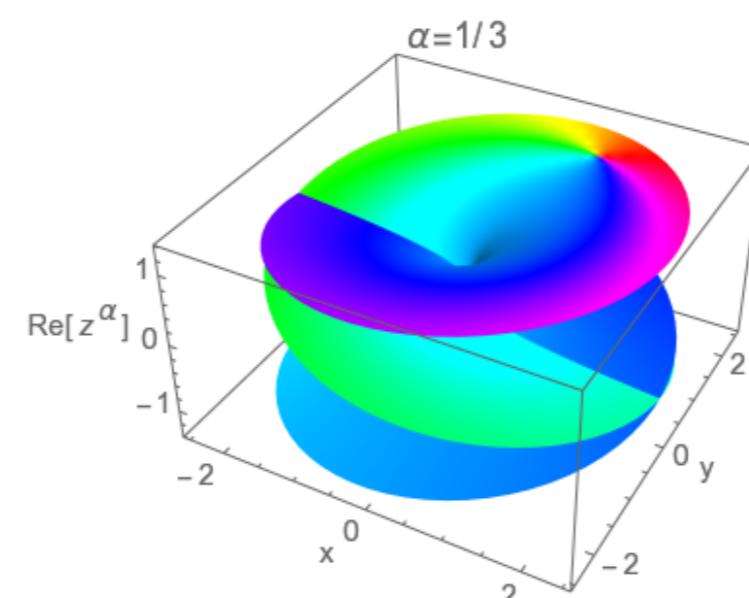
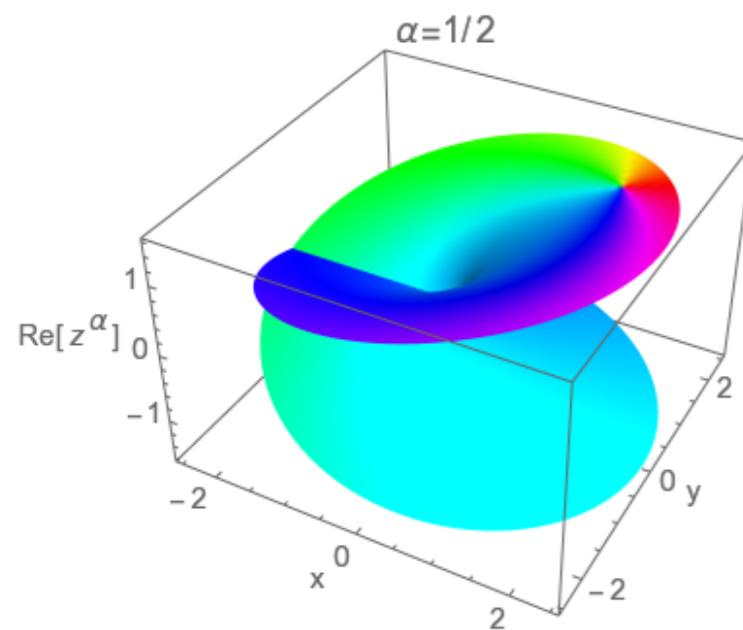


- ◆ The sector may be wrapped onto a cone with opening angle  $\theta_0 = \arcsin(1/\alpha)$
- ◆ A sector of an annulus maps onto a truncated cone:

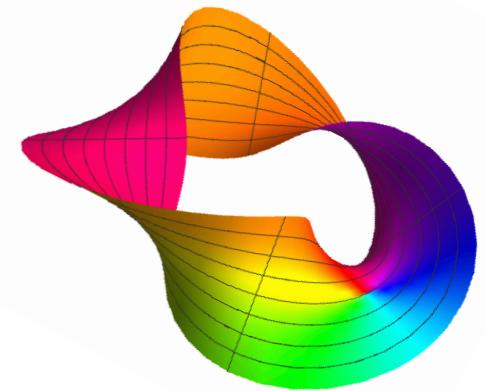
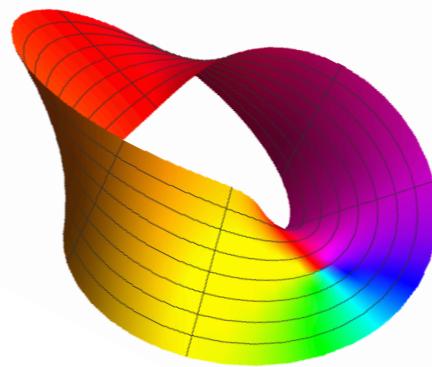
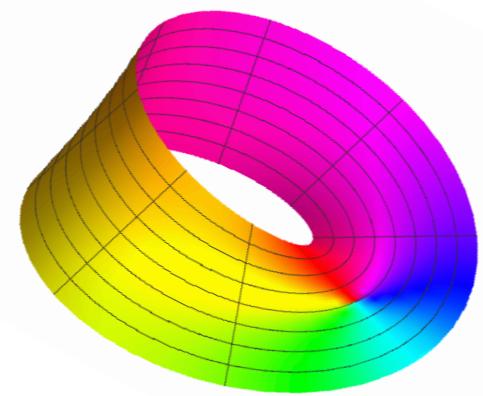
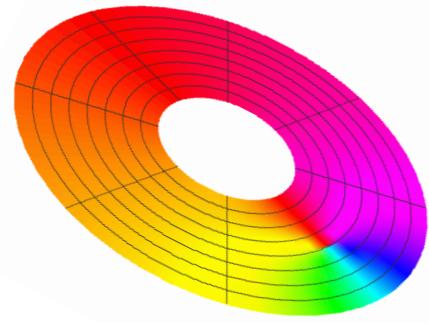


# Riemann surfaces

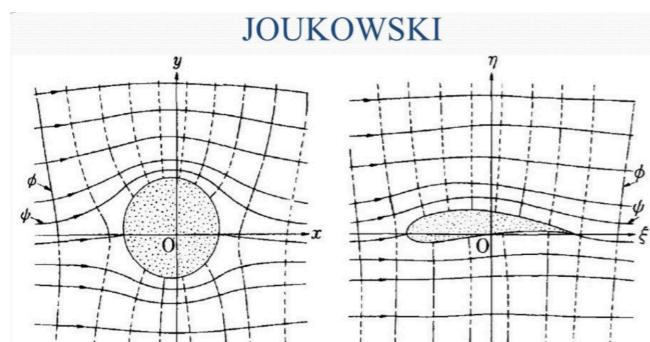
- ◆ The map  $Z = z^\alpha$  is single-valued only for integer  $|\alpha| \geq 1$
- ◆ General  $\alpha$  ?  $0 < \alpha < 1$ ?
- ◆ The function  $\operatorname{Re}(z^{p/q})$  winds  $q$  times around the origin:



# Having fun with Moebius strips

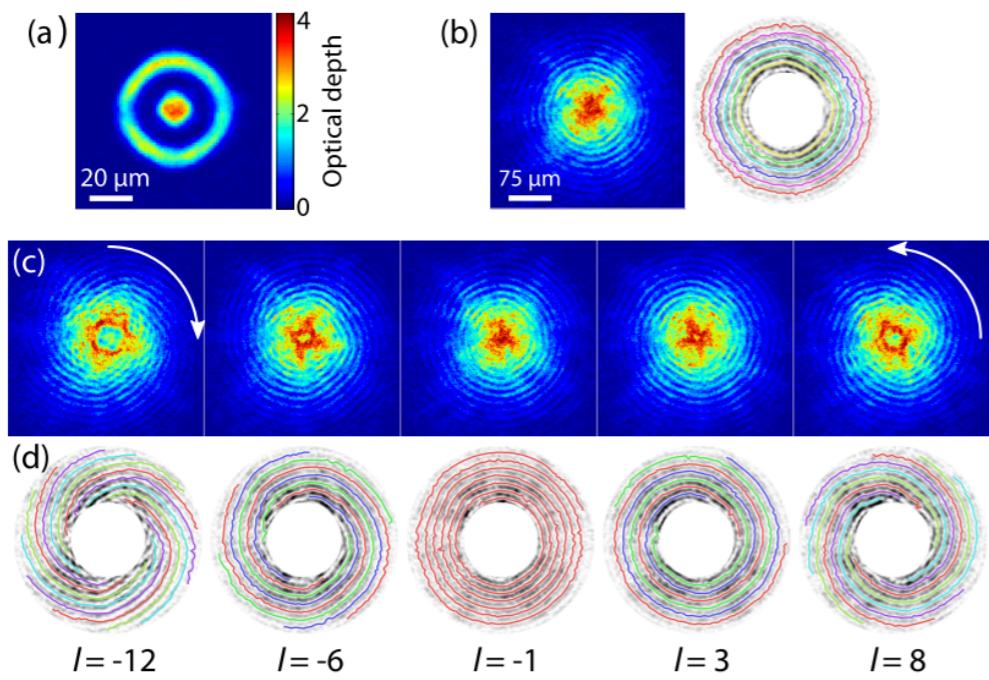


and airplane wings!



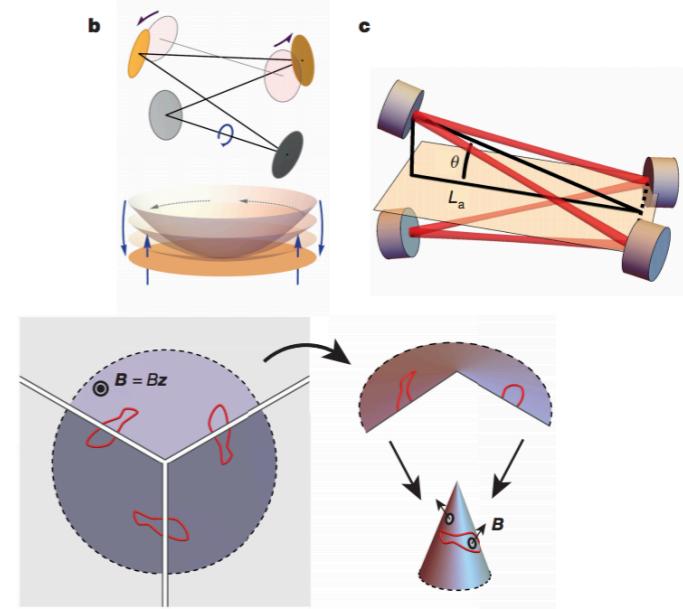
# Experiments?

Ring traps for BECs:



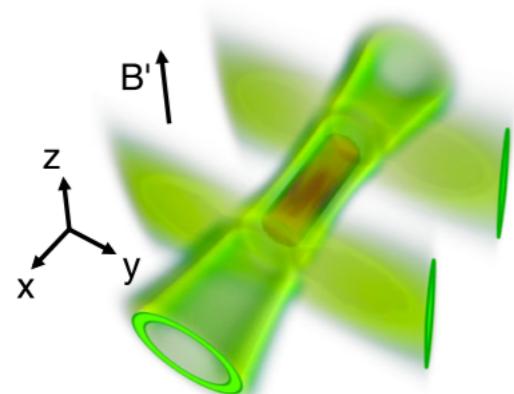
[Eckel *et al.*, Phys. Rev. X (2014)]

Twisted optical cavities:



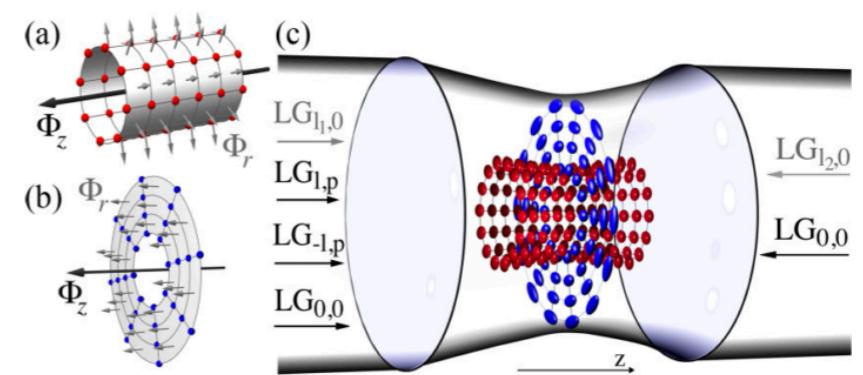
[Schine *et al.*, Nature (2016)]

Cylindrical traps for BECs:



[Gaunt *et al.*, Phys. Rev. Lett. (2013)]

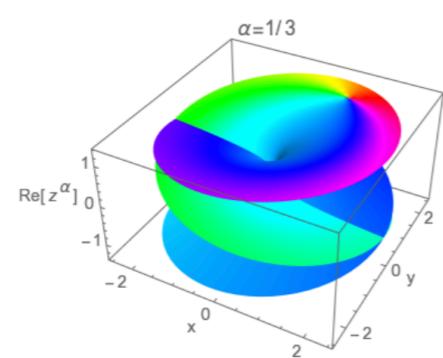
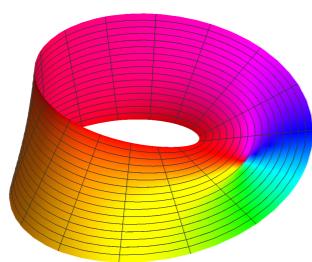
Cylindrical and annular lattices for BECs:



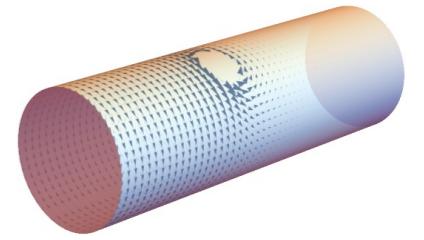
[Łacki *et al.*, Phys. Rev. A (2016)]

# Conclusions

- ◆ *Potential flow theory* describes perfect fluids in 2D
- ◆ Images and conformal maps allow to study peculiar geometries
- ◆ Direct hydrodynamic analog of Laughlin pumping
- ◆ On a cylinder, vortices will not stand still!
- ◆ Single-valuedness of the wave function around the cylinder imposes a quantized translational velocity to the vortex core
- ◆ BKT physics will not happen on a cylinder



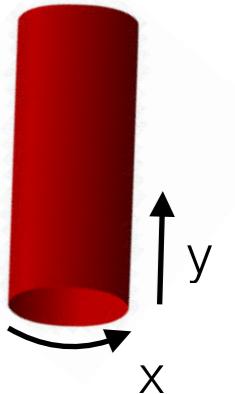
N. Guenther, P. Massignan, and A. Fetter  
Phys. Rev. A **96**, 063608 (2017).



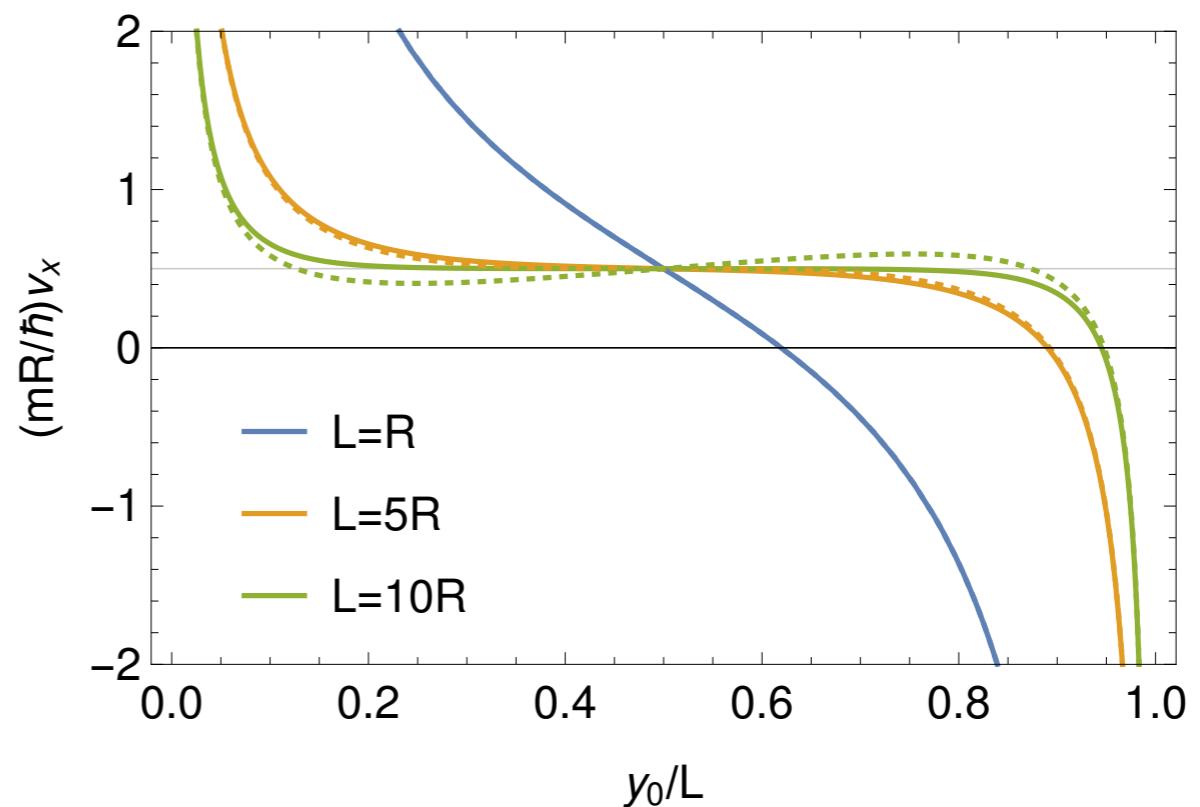
Thank you!

# Vortex on a finite cylinder

- ◆ Cylinder of length  $L$ :  $F_L(z) = \ln \left[ \frac{\vartheta_1\left(\frac{z-z_0}{2R}, e^{-L/R}\right)}{\vartheta_1\left(\frac{z-z_0^*}{2R}, e^{-L/R}\right)} \right]$



- ◆ Vortex velocity:



# Laughlin pumping on a cylinder

- ◆ Start with the fluid at rest
- ◆ Stir the fluid from below at a constant rate
- ◆ A vortex appears on the lower edge, and moves upward
- ◆ The fluid remains stationary above the vortex
- ◆ Below the vortex, the fluid rotates with exactly  $\hbar$  units of angular momentum per particle

